Characterization of Variable Systems

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Outline of the Presentation

- **Objectives**
- **Formalisms for Describing Deterministic Systems**
  - **Operator Formalism**
  - **Generalized Delay Formalism**
  - **Functional Formalism**
  - **Impulse Response Formalism**
  - **Laplace Transform Formalism**
- **Fundamental Theorems**
- **Multidimensional Laplace Transformation (MDLT)**
  - **Bivariate Time and Bifrequency Characterization**
    - **Two-Dimensional Laplace Transform (2DLT)**
    - **Laplace –Carson Transform**
- **Concluding Remarks**
- **References**

Objectives

- **There has been an increasing interest in realization and implementation of variable and adaptive systems.**
- **The main objective of this talk is to provide a unified treatment for the analysis and synthesis of variable systems.**
  - **The theory of nonlinear systems has been widely based on the time-domain approach.**
- **Another objective is to reintroduce the less known MDLT and multidimensional Fourier transform (MDFT) techniques for resolving a system function into its moments and a signal function into its sinusoidal components.**

Fundamental Input-Output Relation

- **A system has a dynamic behavior that is changing with some parameter such as time.**
    - **The *time* is assumed to be *real* for physical systems.**
- **The input-output transformation may depend on various derivatives and/or integrals of both input and output functions.**
- **Under these assumptions, the output vector *y* is related to the input vector *x* (for a SISO system) through an operator, Ω, i.e.,**

$$\Omega(\langle x, y \rangle) = 0$$

Operator Formalism

- **The system operator Ω maps, according to some specified rule, the input functions into the output functions.**

- **The operator Ω is an integrodifferential equation and the auxiliary condition that satisfy the initial condition.**

- **The operator may be classified as homogeneous, linear, deterministic, causal, nonanticipative, etc.**

$$\Omega(\langle x, y\rangle) = \Omega\left(x, \dot{x}, \ddot{x}, \ldots, x^{(p)}; y, \dot{y}, \ddot{y}, \ldots, y^{(q)}\right) = 0,$$

Linear Time Varying Operator

- **A SISO deterministic system operation is**

$$y(t) = \Omega\{x(t)\}$$

- **The system operator is linear *if and only if* the following relation holds:**

$$\Omega[\alpha x_1(t) + \beta x_2(t)] = \alpha y_1(t) + \beta y_2(t)$$

- **The system input can be *any* function including an impulse or a *delta function:***

$$y_\delta(t;\tau) = h(t)\delta(t-\tau)$$

Generalized Delay Formalism

- **The functional input-output relation can be thought of as a composition functional formalism.**
- **A continuous SISO time-invariant variable system, initially at rest, can be <u>symbolically</u> written as:**

$$y(t) = h \circ x(t) = h\big(x(t)\big) = e^{\ln|h(x(t))|}$$

- **Assuming that *dx(t)/dt* is <u>nowhere zero</u> and *x(t)* has a <u>real root</u> at x_0:**

$$y(t) = h_0 e^{\int_{t_0}^{t} \frac{\dot{h}(x(\xi))}{h(x(\xi))} x'(\xi) d\xi} = h_0 e^{\int_{x(t_0)}^{x(t)} g(x) dx}$$

Linear Delay Formalism

- **Let *h(t)* be a (piecewise) continuous function and bounded by a finite number, and its 1st -order and higher-order derivatives exists.**

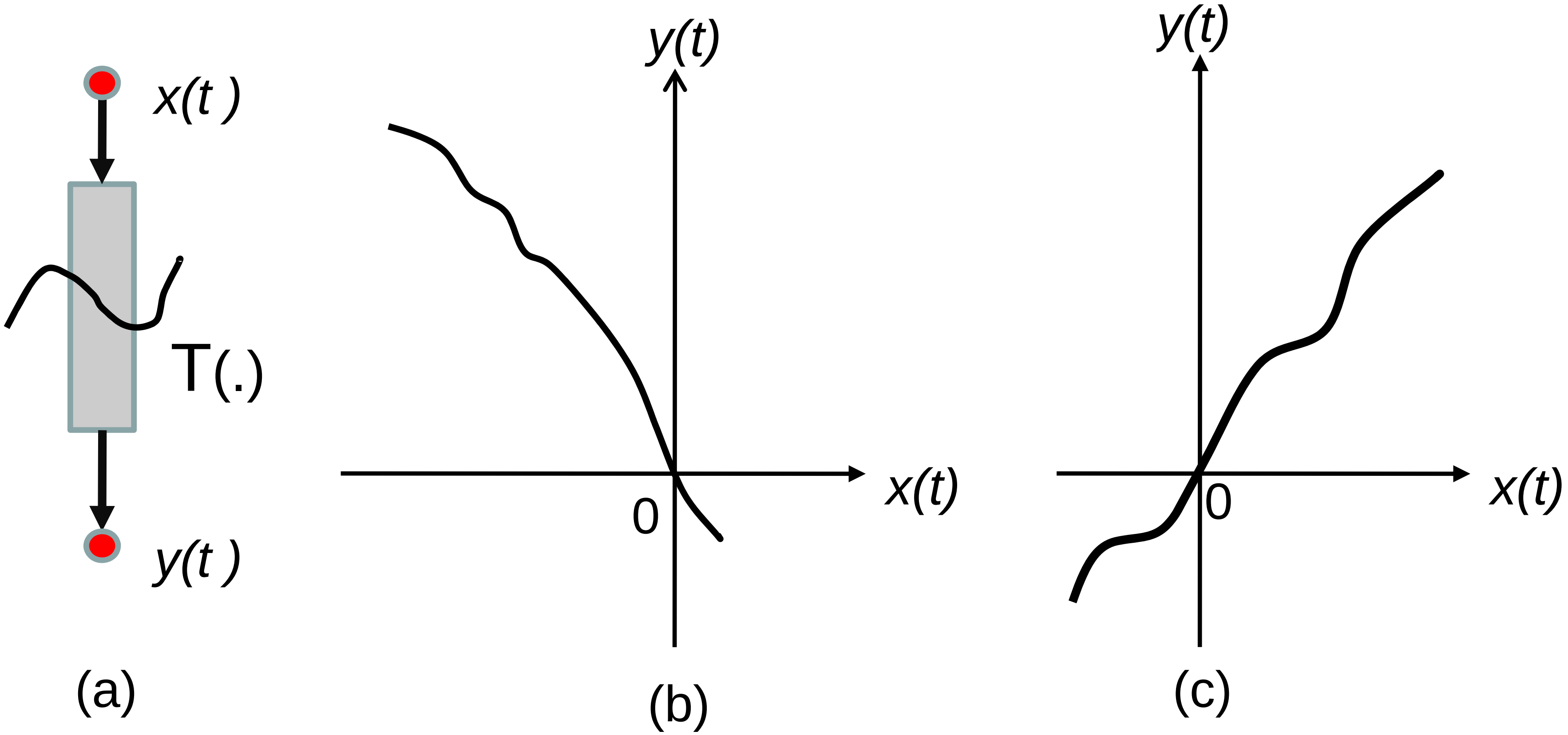
- **The LTV system response can equivalently be written as:**

$$y(t) = e^{-\int_{\tau}^{t} \frac{h'(\xi)d\xi}{h(\xi)}} x(t)$$

- **The system response can be written more compactly and symbolically as a delay multiplier:**

$$y(t) = e^{-g(t,\tau)} x(t)$$

Deterministic Variable System



- ☐ **(a) Block diagram of a variable (sub)system**
- ☐ **(b) Monotonically decreasing response function**
- ☐ **(c) Monotonically increasing response function**

Nonanticipative System Function

- **Define the instant at which the input is applied to the system as the origin for time *"t."***
- **The nonanticipative condition implies that for all admissible input-output pairs, if *x(t)=0*:**

$$y(t) \equiv o \qquad\qquad for \quad t < \tau$$

$$y(t) \equiv T[x(t)] \qquad\qquad for \quad t > \tau$$

- **Then, we may define *T[.]* to be zero for negative values of its argument:**

Impulse Response Formalism

- **The delta function is defined as a *distribution* or a *generalized* function.**
- **The delta function can also be defined as a measure, which accepts as an argument a subset *A* of set *C* and returns *1* if *0* is in *A* and *0* otherwise:**

$$\int_C \delta(|t-\tau|)dt = 1, \quad \tau \in C$$
$$= 0, \quad \tau \notin C$$

- **As a distribution, the composition $\delta(x(t))$ where *x(t)* is a smooth function infinitely differentiable with *dx(t)/dt* nowhere zero will yield:**

$$y(t) = \int_{x(R)} h(\xi)\delta(\xi)d\xi = \int_R h(x(t))\delta(x(t))|\dot{x}(t)|dt$$

Vito Volterra

- **Vito Volterra** (3 May 1860 – 11 October 1940) was an Italian mathematician and physicist, known for his contributions to mathematical biology and integral equations.
- Born in Ancona, then part of the Papal States, into a very poor Jewish family, Volterra showed early promise in mathematics before attending the University of Pisa, where he fell under the influence of Enrico Betti, and where he became professor of rational mechanics in 1883. He immediately started work developing his theory of functionals which led to his interest and later contributions in integral and integro-differential equations. His work is summarised in his book *Theory of functionals and of Integral and Integro-Differential Equations* (1930).

Volterra Functional Formalism (1)

- **Taylor expansion cannot be used to represent nonlinear systems with memory (why?).**

- **Vito Volterra has developed a practical method [Volterra 1930]:**
  - **Let *$H(t,\tau)$=0 represent an algebraic relation,***
  - ***Let the two variables be replaced by two functions $x(t,\tau)$ and $y(t,\tau)$,***
  - ***Let all multiplications of t with itself or with τ be replaced by composition of the corresponding functions as:***

$$x*y=\int x(t,\xi)y(\xi,\tau)d\xi$$

- **Now, it is possible to expand the original function *H(., .)* in power series of t and τ, when they have been replaced by *$x(t,\tau)$* and *$y(t,\tau)$* and multiplications by convolutions.**
- **This will yield the response y(t) as an integral equation!**

Volterra Functional Formalism (2)

- A *functional* is defined as a generalization of a function Γ of several independent variables $x_1(t), x_2(t), \ldots, x_p(t)$.
- The functional $\Gamma(x_1(t), x_2(t), \ldots, x_p(t))$ is a function of the values that the function $x_i(t)$ takes when t lies in some interval .
- The function $x_i(t)$ is arbitrary and so is its independent variable t, i.e., we can write $x_i(t_i)$.
- This definition may readily be extended to the case when there are functions of several variables $x_i(t_1, t_2, \ldots, t_i)$ instead of the function $x_i(t_i)$.
- An *analytic functional* is an infinitely differentiable functional; high-order derivatives approach zero.
- The response of a nonlinear system represented by Γ is some functional of its input function.

Volterra Functional Formalism (3)

Lemma 1 [Parente 1970]– A system *S* is time-invariant deterministic if and only if there exists a functional *H* such that, for all real *t* and each admissible input-output pair,

$$y(t) = H[x(t-\tau)]\Big|_{\tau=t_0}^{t_f}$$

where t_0 and t_f are real constants and $-\infty < t_0 < t_f < \infty$.

- **The closed interval $[t_0, t_f]$ is the system *memory*.**
- **If $t_0 = t_f = 0$, the system is *memoryless*.**
- **If $t_0 = t_f > 0$, the system is a *delay*.**
- **If $t_0 \geq 0$, the system is *causal* and *realizable*.**
- **If $t_0 \leq 0$, the system is *anticipative* and *unrealizable*.**

Volterra Functional Formalism (4)

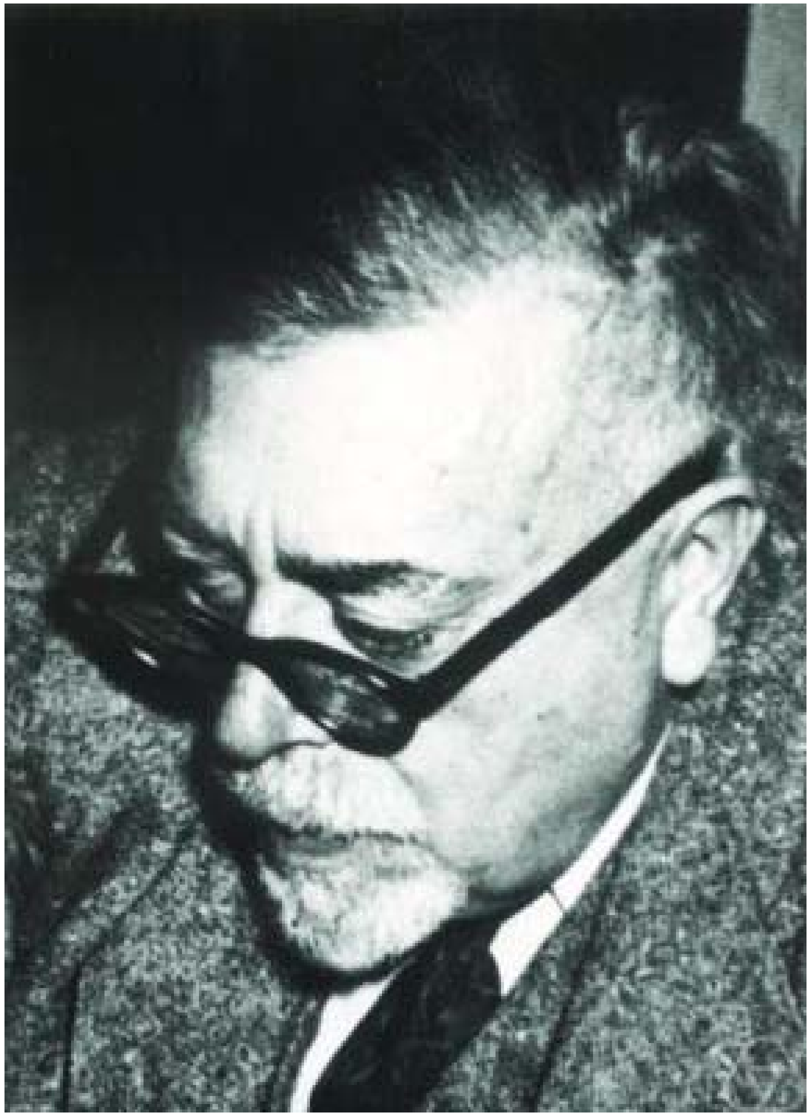
Lemma 2 [Volterra 1930]– If system *S* is a homogenous continuous functional *H* in the field of continuous functions, it can be expanded as a functional power series:

$$H[X(\xi_0)+x(\xi)]\Big|_{\xi=t_0}^{t_f} = H[X(\xi_0)]\Big|_{\tau=t_0}^{t_f} + \int_{t_0}^{t} h_1(\xi_1)x(\xi_1)d\xi_1 + \ldots$$

$$\ldots + \int_{t_0}^{t_f}\int_{t_0}^{t_f}\ldots\int_{t_0}^{t_f} h_n(\xi_1,\xi_2,\xi_3,\ldots,\xi_n)x(\xi_1)x(\xi_2)x(\xi_3)\ldots x(\xi_n)d\xi_1 d\xi_2 d\xi_3\ldots d\xi_n + \ldots$$

- ☐ **This can be used to represent the response as the small-signal changes of the input around an operating point.**

Norbert Wiener



- **Norbert Wiener** (November 26, 1894, Columbia, Missouri – March 18, 1964, Stockholm, Sweden) was an American mathematician.
- A famous child prodigy, Wiener (*pronounced WEE-nur*) later became an early studier of stochastic and noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems.
- Wiener is regarded as the originator of cybernetics, a formalization of the notion of feedback, with many implications for engineering, systems control, computer science, biology, philosophy, and the organization of society.

Volterra Functional Formalism (5)

Theorem 1 [Wiener 1942, Parente 1966]– A SISO system *S* is a analytic if and only if there exists a Volterra functional series such that for all real time *t* and each admissible input-output pair,

$$y(t) = h_0(t_0) + \int_{-\infty}^{\infty} h_1(t_1)x(t-t_1)dt_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(t_1,t_2)x(t-t_1)x(t-t_2)dt_1dt_1 + \dots$$

$$y(t) = \lim_{n\to\infty}\sum_{i=0}^{n}\int_{t_0}^{t_f}\int_{t_0}^{t_f}\dots\int_{t_0}^{t_f} h_i(\xi_1,\xi_2,\xi_3,\dots,\xi_i)\prod_{i}[x(t-t_i)d\xi_i]$$

- Each term is a convolution and is called the homogenous functional of the *i*-th degree.
- The kernels are not unique, but can be found uniquely if assumed to be symmetric and related to the functional derivatives.
- If the input function is taken to be a delta function, this gives the impulse response.
- The *i*-th Volterra kernel of the system, h_i, is the *i*-th order impulse response function of $t_1, t_2, t_3, \dots, t_i$ real variables.

Volterra Functional Formalism (6)

Theorem 2 [Erfani 2010]– A SISO system *S* is a causal autonomous deterministic system if and only if there exists a functional *H* such that for all real time *t* and τ and each admissible input-output pair,

$$y(t) = H[x(|t-\tau|)]\,\Big|_{\tau=t_0}^{t_f} \;, \qquad \forall t_0 \le \tau \le t \le t_f$$

where the closed interval [t_0, t_f] might include infinity, and |.| denotes a *norm* (or *length*) of the variable (or function) inside the bar signs.

- **Definition - A *causal function* is defined as a function of a *norm* of the time vector *t*.**

Special Case: Linear Time-Varying Systems

- ☐ For homogeneous linear time-invariant (LTI) systems, all Volterra kernels, except $h_1(t_1)$ are identically zero [Mitzel 1977].

- ☐ For homogeneous linear time-Varying (LTV) systems, the *2-nd* degree kernel, $h_2(t_1, t_2)$, also exists [Erfani 2009].

$$y(t) = \int_R h_2(\tau, t) x(\tau) d\tau , \quad t \geq 0$$

- ☐ The 1-st degree kernel, $h_1(t_1)$, is the impulse response of $h_2(t_1, t_2)$, and is called the transient-response of an initially relaxed LTV systems:

$$y_\delta(t) = \int_R h_2(\tau, t) \delta(| t - \tau |) d\tau = h_2(\tau, t) |_{t=\tau} \equiv h_1(\tau), \quad \tau \geq 0$$

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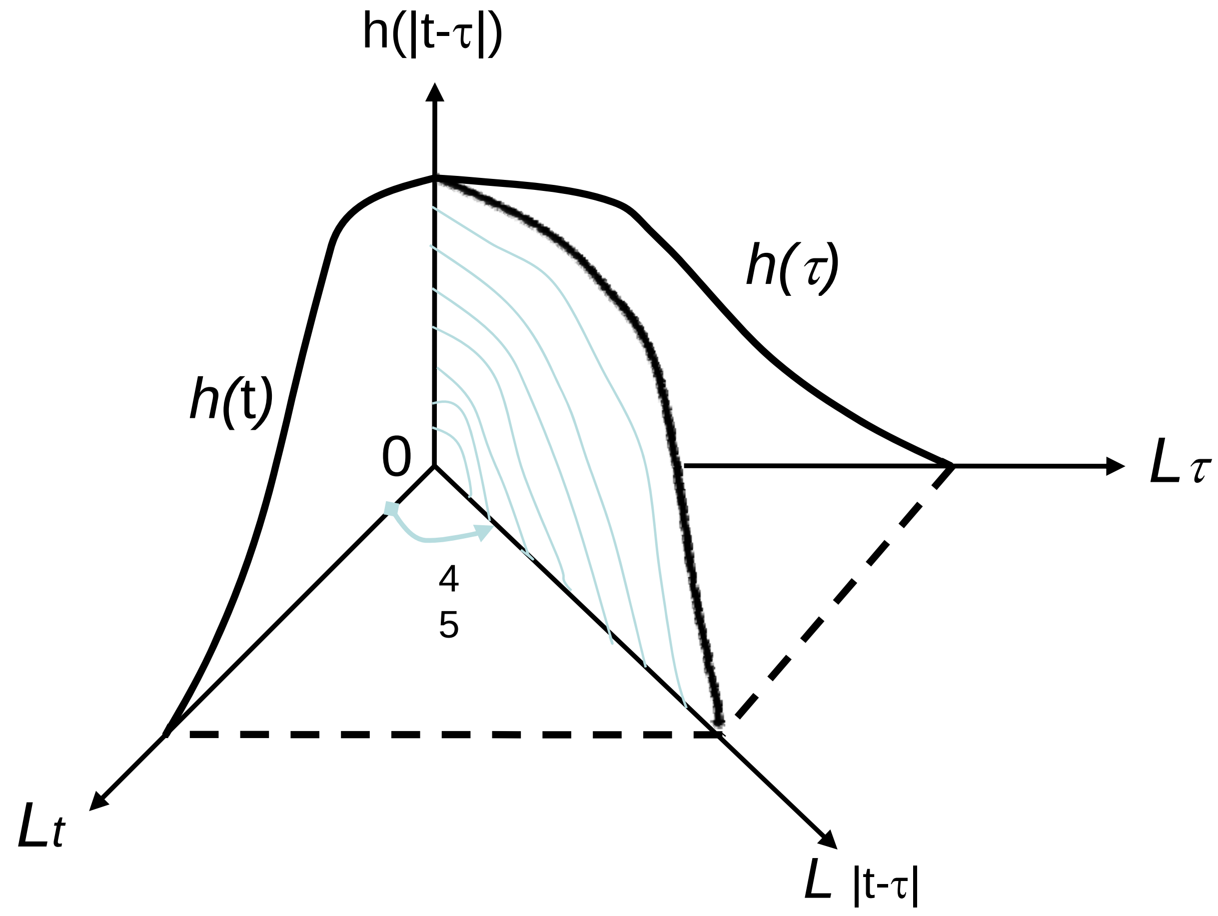
L. A. Zadeh

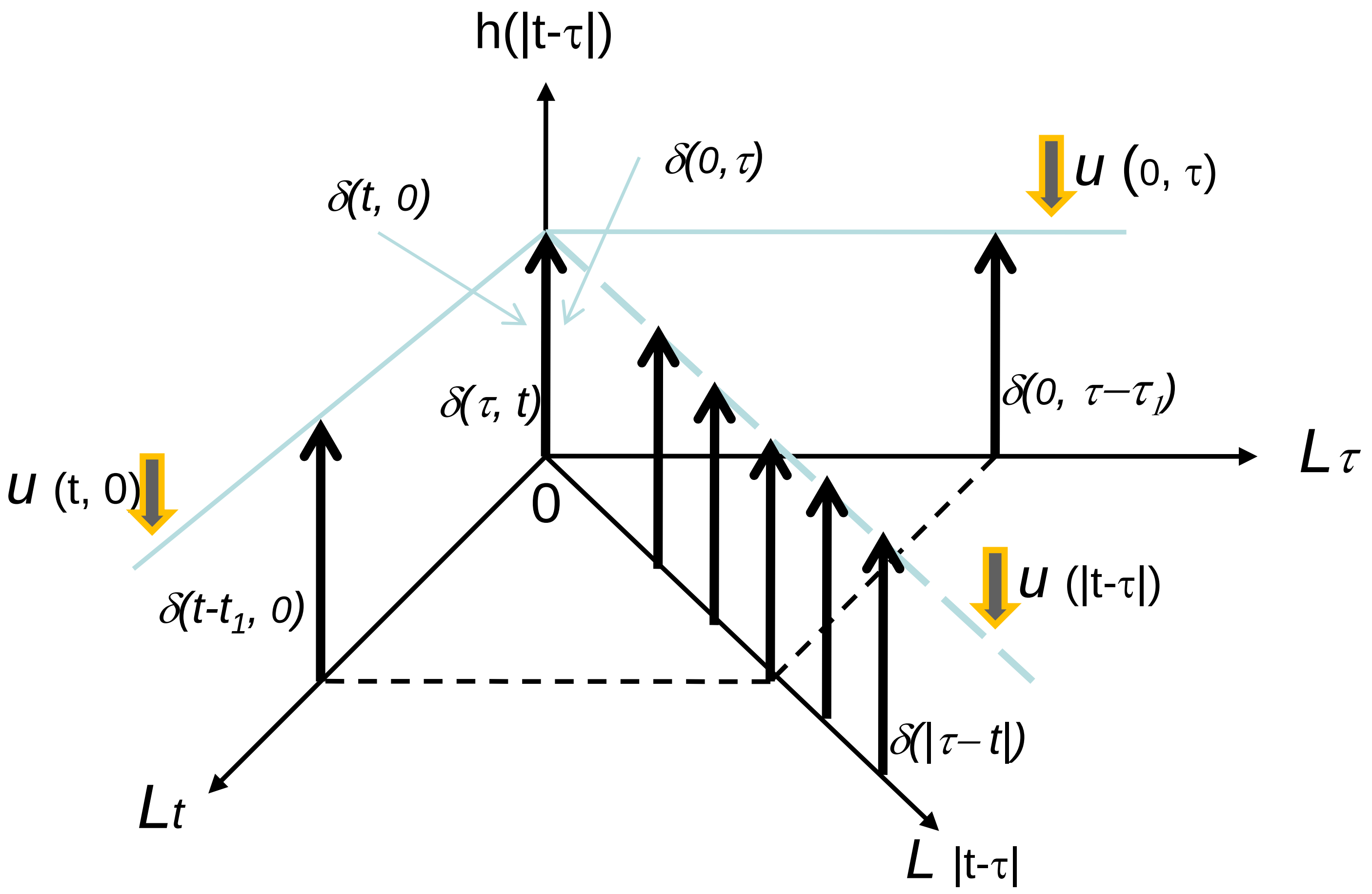
- **Lotfali Askar-Zadeh** (born February 4, 1921), is a mathematician, electrical engineer, computer scientist, and a professor of computer science at the University of California, Berkeley.
- Zadeh was born in Baku, Azerbaijan SSR, to an Iranian Azeri father and a Russian Jewish mother. At the age of ten the Zadeh family moved to Iran.
- In 1942, he graduated from the University of Tehran with a degree in electrical engineering (Fanni), and moved to the United States in 1944. He received an MS degree in electrical engineering from MIT in 1946, and a PhD in electrical engineering from Columbia University in 1949.
- Zadeh taught for ten years at Columbia University, was promoted to Full Professor in 1957, and has taught at the University of California, Berkeley since 1959. He published his seminal work on fuzzy sets in 1965, in which he detailed the mathematics of fuzzy set theory. In 1973 he proposed his theory of fuzzy logic.

Profile of Bivariate Functions



- ☐ **Representation of a *2-nd* degree impulse response function; line $L_{|t-\tau|}$ denotes a "zero" for function *h(|t-τ|)*.**

Two-Dimensional Delta Functions



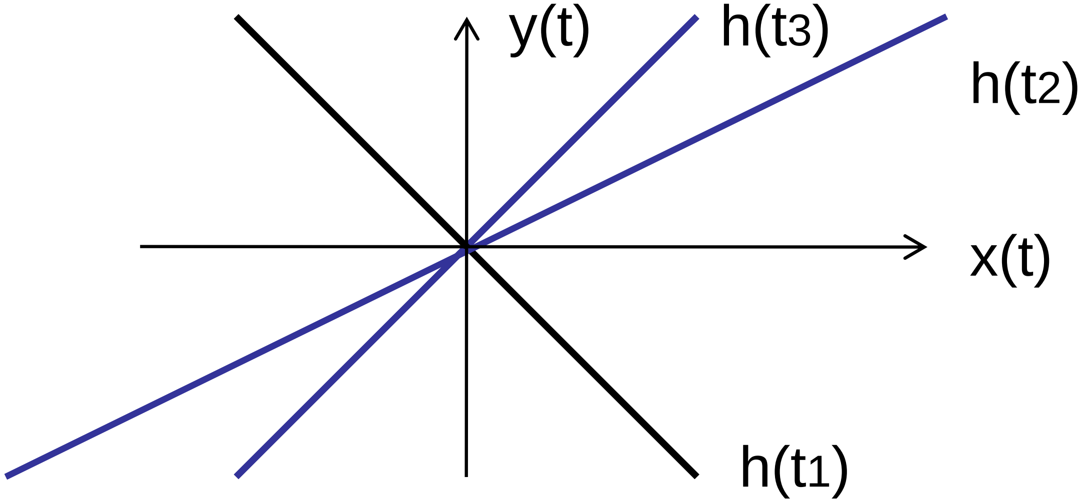
- **The 2D unit-impulse functions used to determine a causal deterministic LTV function as a function of the l_2 norm of the complex quantity $t + j\tau$.**

Linear Time Varying Elements

- **A single-input single-output (SISO) dynamic system element (e.g., a resistor, capacitor, or inductor) of finite order characterized by its input-output relationship is said to be linear if the following holds for each t, τ ≥0:**

$$y(t;\tau)=h_2(|\tau-t|)x(t;\tau)$$

  - **Where *h(t-τ)* is the system function defines the response at time t, denotes the slope of the y-x curve in a rectangular coordinates system.**



Transform Formalism

- **Multidimensional (two-sided) Laplace transform (MDLT) techniques can be used to transform the analytical variable system into the frequency domain.**
- **Taking MDLT from *nontrivial* terms of the Volterra-Wiener functional formalism, with some simplifying assumptions, we can write:**

$$Y(s_1, s_2, s_3, \ldots, s_i) = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{i!} Y_i(s_1, s_2, s_3, \ldots, s_i) =$$

$$\lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{i!} H_i(s_1, s_2, s_3, \ldots, s_i) X_1(s_1) X_2(s_2) \ldots X_i(s_i)$$

- **Symbolically, this equation can be written in a compact form as:**

$$\boldsymbol{Y}(\boldsymbol{s}) = \boldsymbol{H}(\boldsymbol{s}) \boldsymbol{X}(\boldsymbol{s})$$

Matrix function of vector **s**

- **To convert the above MDLT function into a single frequency s, the technique of *association of variables* is used [Chen 1973].**

Pierre-Simon de Laplace

Pierre-Simon, marquis de Laplace (23 March 1749 – 5 March 1827) was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics. He summarized and extended the work of his predecessors in his five volume *Mécanique Céleste* (Celestial Mechanics) (1799–1825). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the so-called Bayesian interpretation of probability was mainly developed by Laplace.[1]

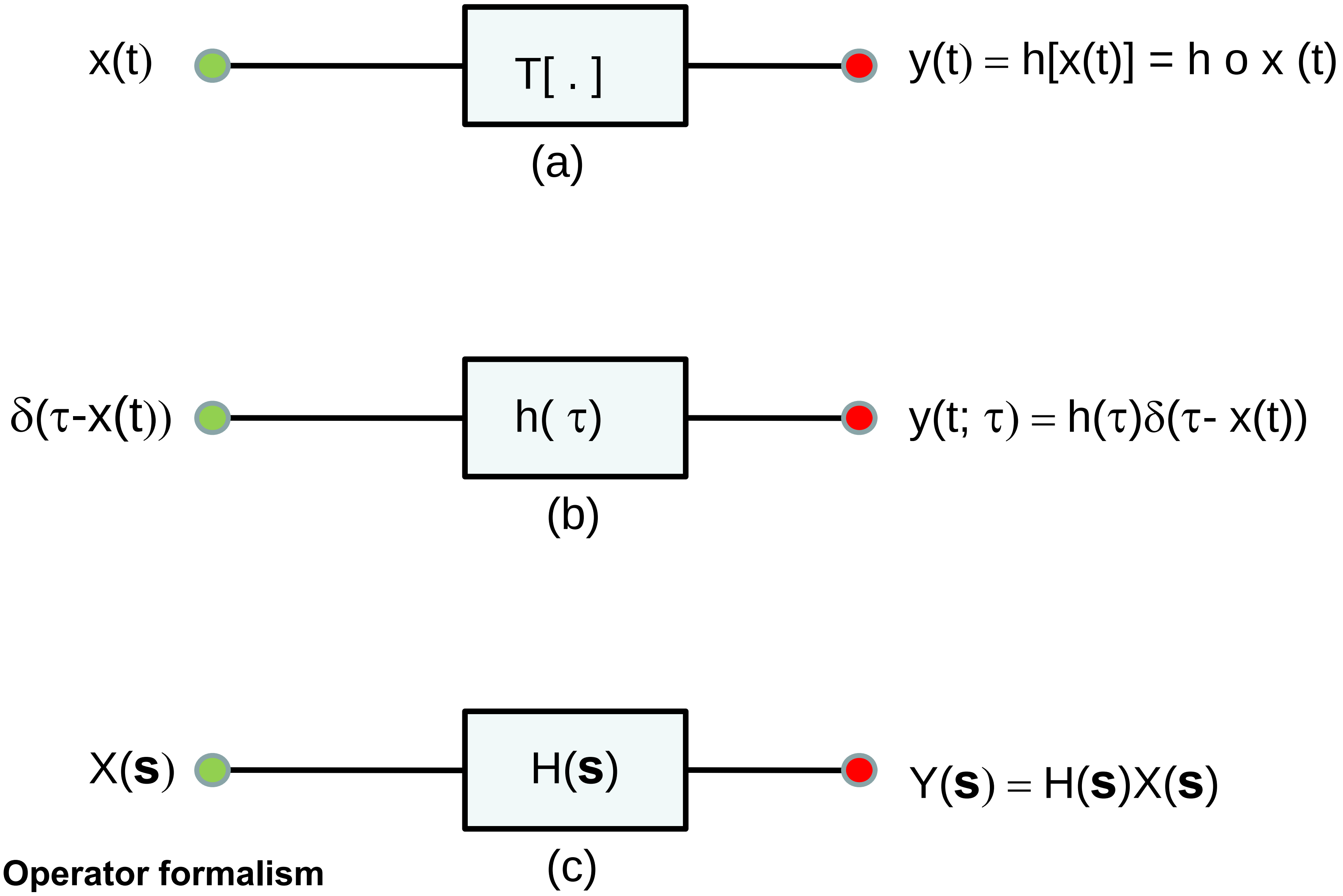
He formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming. The Laplacian differential operator, widely used in applied mathematics, is also named after him.

He restated and developed the nebular hypothesis of the origin of the solar system and was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.

He is remembered as one of the greatest scientists of all time, sometimes referred to as a *French Newton* or *Newton of France*, with a phenomenal natural mathematical faculty superior to any of his contemporaries.[2]

He became a count of the First French Empire in 1806 and was named a marquis in 1817, after the Bourbon Restoration.

Block Diagram Representation



$x(t)$ → $T[\,.\,]$ → $y(t) = h[x(t)] = h \circ x\,(t)$

(a)

$\delta(\tau - x(t))$ → $h(\tau)$ → $y(t;\tau) = h(\tau)\delta(\tau - x(t))$

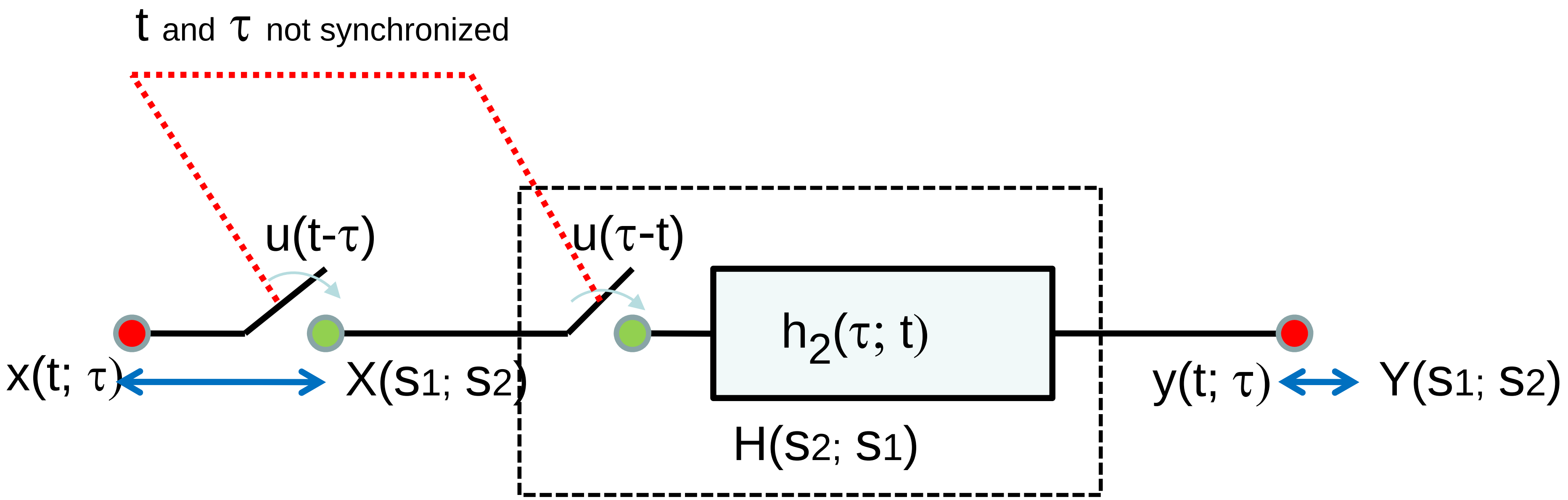
(b)

$X(\mathbf{s})$ → $H(\mathbf{s})$ → $Y(\mathbf{s}) = H(\mathbf{s})X(\mathbf{s})$

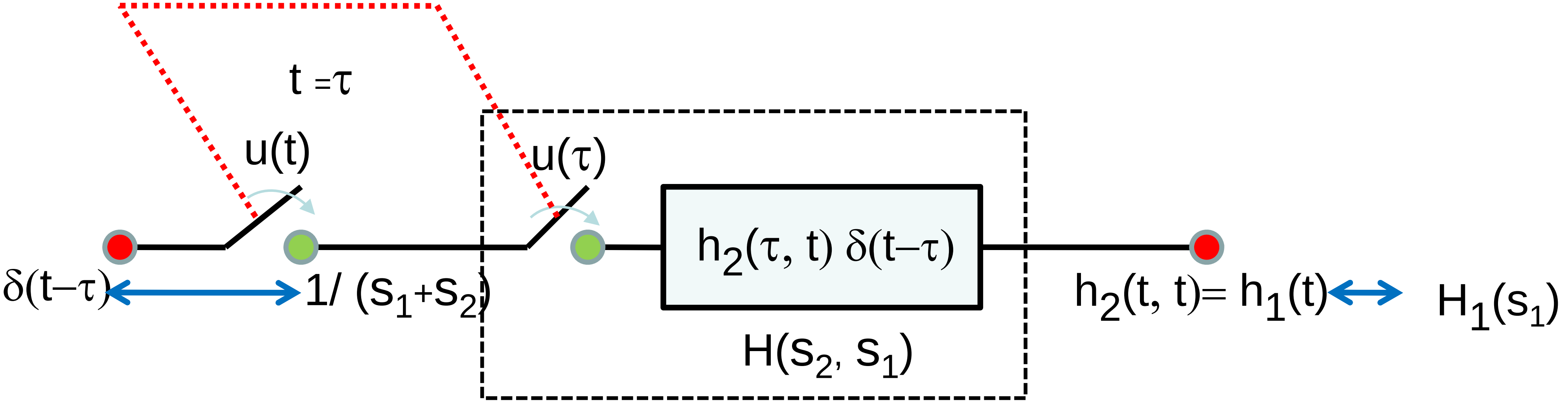
(c)

- **a) Operator formalism**
- **b) Impulse response formalism**
- **c) Transform formalism**

The 2nd and 1st Degree Impulse Responses



(a)



(b)

Multi-Dimensional Laplace Transform (MDLT)

- ❑ **MDLT pairs can be employed to obtain a frequency-domain formalism**

$$h(\vec{t}) \Leftrightarrow H(\vec{s}) = \int_0^\infty \int_0^\infty \ldots \int_o^\infty h(\vec{t})e^{-\vec{s}.\vec{t}} \prod_{i=1}^{n} dt_i$$

$$H(\vec{s}) \Leftrightarrow h(\vec{t}) = \left(\frac{1}{2\pi j}\right)^n \int_{\sigma_n - J\infty}^{\sigma_n + J\infty} \int_{\sigma_{n-1} - J\infty}^{\sigma_{n-1} + J\infty} \ldots \int_{\sigma_1 - J\infty}^{\sigma_1 + J\infty} H(\vec{s})e^{\vec{s}.\vec{t}} \prod_{i=1}^{n} ds_i$$

where

$$\vec{t} = (t_1, t_2, \ldots, t_i)$$

$$\vec{s} = (s_1, s_2, \ldots, s_i)$$

$$\vec{s} \cdot \vec{t} = \sum_{i=1}^{n} s_i t_i$$

Inner Product

Two-Dimensional Laplace Transform (2DLT)

- ❑ **For conformal transformation, it is required that the unit function u(t, т) transforms into itself:**

$$u(t,\tau) \Leftrightarrow U(s_1,s_2) = 1$$

u (t,т) is equal to 1 when both t and т are positive, and is equal to zero when at least one of the arguments is negative.

- ❑ **Based on the above observation we modify 2DLT as:**

$$h(t,\tau) \Leftrightarrow H(s_1,s_2) = s_1 s_2 \int_0^{+\infty}\int_0^{+\infty} h(t,\tau) e^{-s_1 t} e^{-s_2 \tau} dt d\tau$$

Laplace-Carson Transform

Example 1 - Laplace Transform of the Impulse Function

- The ordinary unilateral Laplace transform of $\delta(t-\tau)$ is obtained as:

$$L\{\delta(t-\tau)\} = \int_{0-}^{+\infty} \delta(t-\tau) e^{-s_1 t} dt = e^{-s_1 \tau}$$

- This is a function of the variable application time *τ*.

- A second transformation yields:

2DLT

$$L_{2D}\{\delta(t-\tau)\} = \int_{0-}^{+\infty} e^{-s_1 \tau} e^{-s_2 \tau} d\tau = \frac{1}{s_1 + s_2}$$

Example 2 – Laplace-Carson Transform of 2D Impulse

- **The unit–impulse function δ(t, τ) is defined as:**

$$\delta(t,\tau)=\delta(t)\delta(\tau)$$

The Laplace-Carson transform of δ(t-τ) is

$$\Delta(s_1,s_2)=s_1s_2\int_0^{+\infty}\int_0^{+\infty}\delta(t-\tau)e^{-s_1t}e^{-s_2\tau}dtd\tau=\frac{s_1s_2}{s_1+s_2}$$

Similarly, the L-C transform of δ(t,τ) is $\delta(t,\tau)\Leftrightarrow s_1s_2$

- **We can obtain the L-C transform of** $h(t)\delta(t-\tau)$**as:**

$$h(t)\delta(t-\tau)\Leftrightarrow s_1s_2\int_0^{+\infty}h(\tau)e^{-(s_1+s_2)\tau}d\tau=s_1s_2H(s_1+s_2)$$

Example 3- Two-Dimensional Step Function

- ☐ **The unit–step function u(t, τ) is defined as:**

$$u(t,\tau) \Leftrightarrow U(s_1, s_2) = 1$$

- ☐ **The *Laplace-Carson* transform of u(t-τ) is**

$$U(s_1, s_2) = s_1 s_2 \int_0^{+\infty} \int_0^{+\infty} u(t-\tau) e^{-s_1 t} e^{-s_2 \tau} dt d\tau = \frac{s_2}{s_1 + s_2}$$

Similarly, the L-C transform of u(τ-t) is $\frac{s_1}{s_1 + s_2}$

- ☐ **What is the Laplace-Carson transform of the following?**

$$u(t-\tau)u(\tau-t) = \begin{cases} 1 & t = \tau \\ 0 & t \neq \tau \end{cases}$$

Example 4 - Frequency-Domain Representation of Nonanticipative System Elements

- **Let us define:**

$$h_1(t,\tau)=\begin{cases} h(t-\tau) & for \quad t>\tau \\ 0 & for \quad t<\tau \end{cases}$$

- **The 2DLT is:**

$$H_1(s_1,s_2)=\int_0^{+\infty} e^{-s_2\tau}d\tau\int_\tau^{+\infty} e^{-s_1 t}h(t-\tau)dt = H(s_1)\Big/ s_1+s_2$$

- **Similarly, we define :**

$$h_2(t,\tau)=\begin{cases} h(\tau-t) & for \quad \tau>t \\ 0 & for \quad \tau<t \end{cases}$$

$$H_2(s_1,s_2)=H(s_2)\Big/ s_1+s_2$$

2DLT

- **Adding together, we obtain:**

$$L_{2D}\{h(|t-\tau|)\}=H(s_1,s_2)=\frac{H(s_1)+H(s_2)}{s_1+s_2}$$

Example 5 -The 2DLT of General LTV Systems (1)

- **Consider a SISO LTV system, initially at rest, described by:**

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t)}{dt^i} = \sum_{k=0}^{m} b_k(t)\frac{d^k x(t)}{dt^k}$$

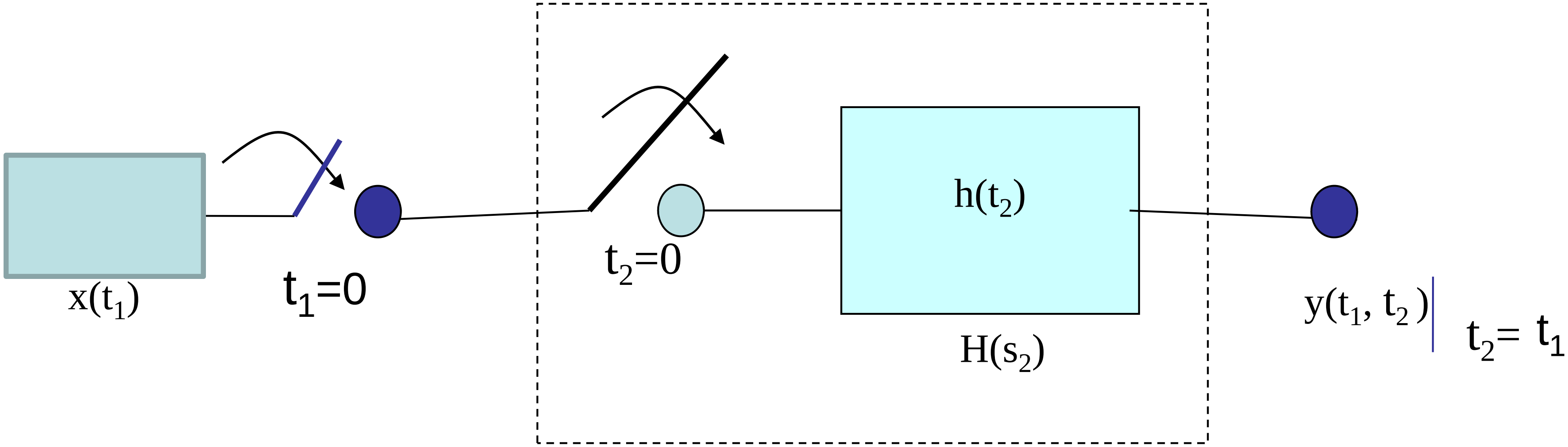
- **For causal inputs *x(.)* and initially relaxed system, the 2D delta function $\delta(t,\tau)=\delta(t)\,\delta(\tau)$ is applied to the system,**

$$\sum_{i=0}^{n} a_i(\tau)\frac{d^i y(t,\tau)}{dt^i} = \sum_{k=0}^{m} b_k(\tau)\frac{d^k \delta(t)}{dt^k}\delta(\tau)$$

- **Taking the 2DLT, we obtain:**

$$H(s_1,s_2) = \frac{\sum_{k=0}^{m} B_k(s_2)s_1^k}{\sum_{i=0}^{n} A_i(s_2)s_1^i}$$

Example 5 - LTI System Is Equivalent to a LTV Synchronized System (2)

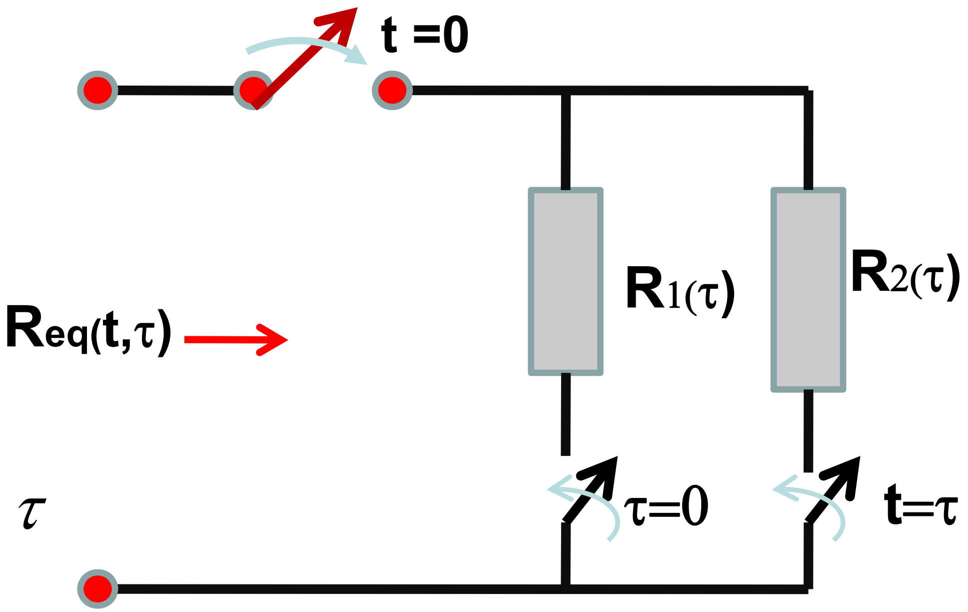


***t_1* is the <u>observation time of</u> signal and *t_2* is the <u>application time to</u> the system.**

Example 6 – Linear Modulation

- **The equivalent resistance of a parallel combination of two LTV resistors in the time-domain is given as:**

$$R_{eq}(t,\tau)=\begin{cases} R_1(\tau-t) & t<\tau \\ R_1(t-\tau)\,||\,R_2(t-\tau) & t\geq\tau \end{cases}$$

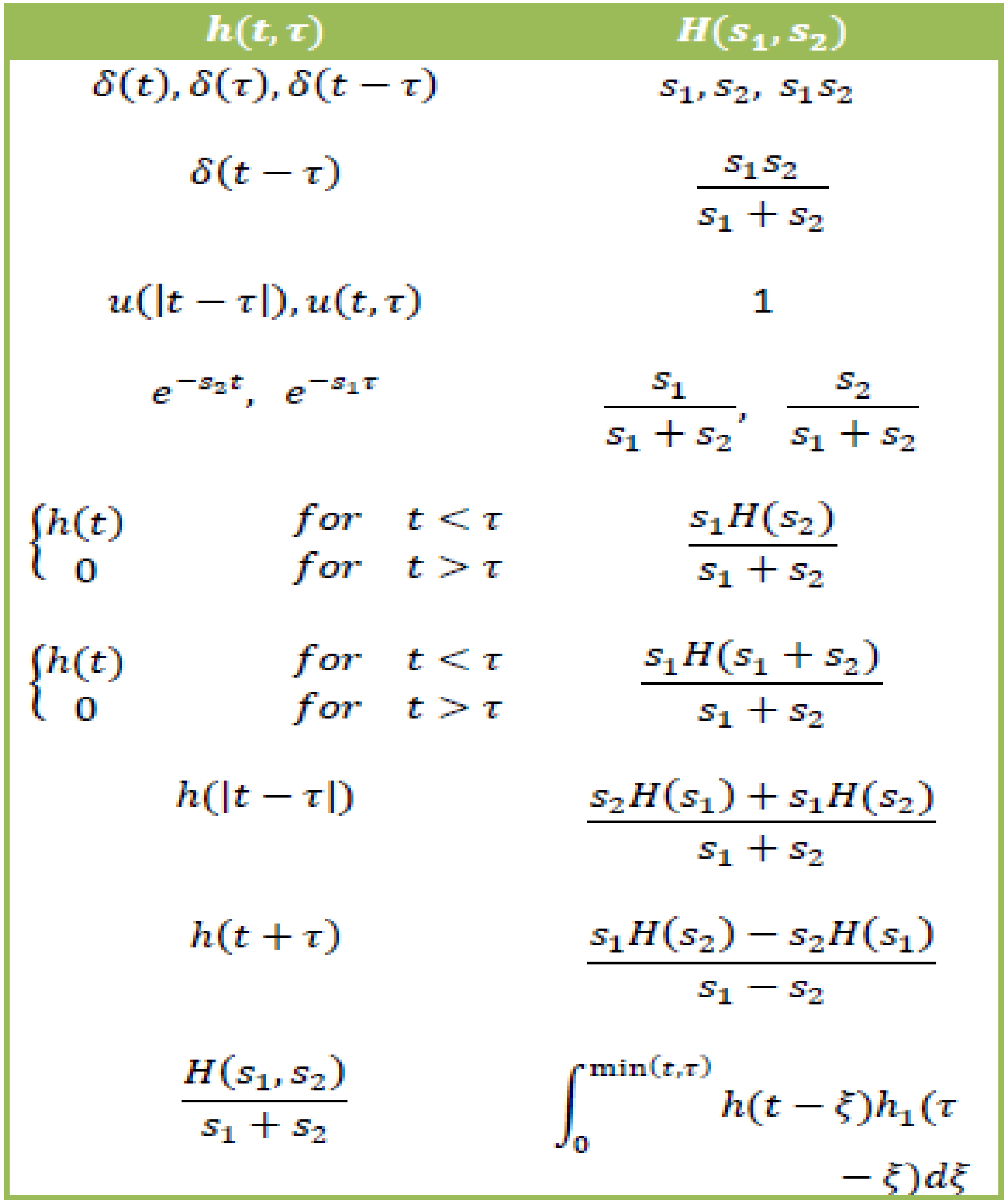


- **The equivalent resistance in the frequency-domain, using the 2DLT, is obtained as:**

$$R_{eq}(s_1,s_2)=\begin{cases} \dfrac{s_1}{s_1+s_2}R_1(s_2) & t<\tau \\ \dfrac{s_1}{s_1+s_2}\dfrac{R_1(s_2)R_2(s_2)}{R_1(s_2)+R_2(s_2)} & t\geq\tau \end{cases}$$

- **The equivalent resistance is directly mapped into the *bifrequency-plane*, subject to an extra multiplication by factor** $\dfrac{s_1}{s_1+s_2} \Leftrightarrow e^{-s_2 t}$ **. (why?)**

2DLT Fundamental Transform Relations



$h(t,\tau)$	$H(s_1,s_2)$
$\delta(t), \delta(\tau), \delta(t-\tau)$	$s_1, s_2,\ s_1 s_2$
$\delta(t-\tau)$	$\dfrac{s_1 s_2}{s_1+s_2}$
$u(\lvert t-\tau\rvert), u(t,\tau)$	1
$e^{-s_2 t},\ \ e^{-s_1 \tau}$	$\dfrac{s_1}{s_1+s_2},\ \ \dfrac{s_2}{s_1+s_2}$
$\begin{cases} h(t) & for \quad t<\tau \\ 0 & for \quad t>\tau \end{cases}$	$\dfrac{s_1 H(s_2)}{s_1+s_2}$
$\begin{cases} h(t) & for \quad t<\tau \\ 0 & for \quad t>\tau \end{cases}$	$\dfrac{s_1 H(s_1+s_2)}{s_1+s_2}$
$h(\lvert t-\tau\rvert)$	$\dfrac{s_2 H(s_1)+s_1 H(s_2)}{s_1+s_2}$
$h(t+\tau)$	$\dfrac{s_1 H(s_2)-s_2 H(s_1)}{s_1-s_2}$
$\dfrac{H(s_1,s_2)}{s_1+s_2}$	$\int_0^{\min(t,\tau)} h(t-\xi)h_1(\tau-\xi)\,d\xi$

a

Conclusions

- **A variable system can be characterized by various formalisms.**
- **The MDLT can be used as an operational calculus for system characterization, especially, for analog signal processing problems.**
- **The transform approach allows, in effect, MDLT techniques to be used for variable systems in the same manner that the conventional frequency-domain techniques are used in connection with fixed systems.**
- **Using MDLT techniques, a *variable system* as well as a *LTV system*, which are described by *partial differential equations* and *ordinary differential equations*, respectively, can be transformed into algebraic polynomial equations of *two* or *more* variables, and easily be solved.**
- **The work presented here opens several areas in the theory of variable systems for further investigations.**

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Questions?