

Are Time-Varying Systems Laplace-Transformable?

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Outline of the Presentation

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Objectives

- ❖ There has been an increasing interest in realization and implementation of linear time-varying (LTV) and adaptive systems.
- ❖ The rather limited use of LTV networks in analog signal processing and communication fields is due largely to lack of powerful means for their analysis, synthesis and performance evaluation.
- ❖ The theory of LTV systems has been widely based on the time-domain approach.
- ❖ The main objective of this project is to provide a unified treatment for the analysis and synthesis of LTV systems.
- ❖ The approach to be described consists of extension of Laplace transform techniques commonly used for linear time-invariant (LTI) systems.

Fundamental Input-Output Representation

- **The classical theory of variable systems is based on the solutions of linear ordinary differential equations with varying coefficients.**
 - The varying coefficients are functions of an independent variable, conveniently called the *time*.
 - The *time* is assumed to be *real* for physical systems.

$$\sum_{i=0}^n a_i(t) y^{(i)}(t) = \sum_{k=0}^m b_k(t) x^{(k)}(t)$$

Observations on LTV Dynamic systems

Observation 1 – In general, time variables of the *signal* and *system* do not have to be synchronized; i.e., the (time) variables of the signal and system are *independent* of each other.

$$L(D, \tau)y(t) = K(D, \tau)x(t)$$

Observation 2 – At any instant of “ t ” there is a response, which is a specified function of “ T ”.

Observation 3 – At any fixed “ T ” there is a response, which is a specified function of “ t ”.

Observation 4 – The system response is a function of variations of *observation* parameter “ t ” and *application* parameter “ T ”.

Observation 5 – A zero-input, SISO LTV system described by:

$$L(D, \tau)y(\cdot) = 0$$

is a linear system that its natural frequencies are varying with “ T ”. In other words, solutions of this equation are exponential functions of time with varying natural frequencies, as given by:

$$y(\cdot) = \sum_{i=0}^n c_i e^{-t\alpha_i(\tau)}$$

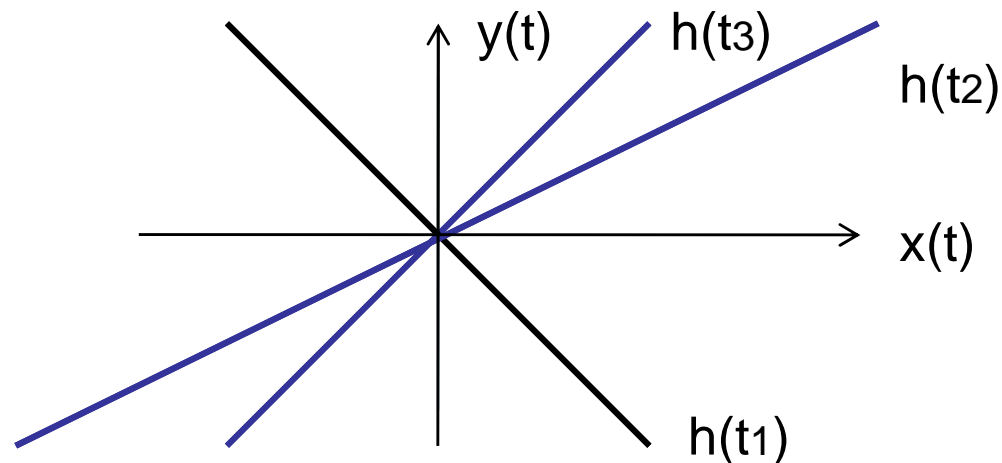
where $\alpha_i(\tau)$ is a function of variable coefficients of the fundamental equation of the system under consideration.

Linear Time Varying Elements

- A single-input single-output (SISO) dynamic system element of finite order characterized by its input-output relationship is said to be linear if the following holds for each $t \geq 0$:

$$y(t) = h(t)x(t)$$

- Where $h(t)$ is the system function defines the response at time t , denotes the slope of the y - x curve in a rectangular coordinates system.



Linear Time Varying Operator

- A SISO dynamic system operation is shown symbolically by:

$$y(t) = \mathcal{O}\{x(t)\}$$

- The system operator is linear if and only if the following relation holds:

$$\mathcal{O}\{\alpha x_1(t) + \beta x_2(t)\} = \alpha \mathcal{O}\{x_1(t)\} + \beta \mathcal{O}\{x_2(t)\} = \alpha y_1(t) + \beta y_2(t)$$

- The system input can be **any** function including an impulse or a *delta function*:

$$y_{\delta}(t; \tau) = h(t) \delta(t - \tau)$$

Observing an Impulse Response Function (1)

- **Observation 6** – The following symbolic identity holds:

$$h(t)\delta(t-\tau) = h(\tau)\delta(\tau-t)$$

- **Observation 7** - The product $h(t)\delta(t-\tau)$ is different from zero at the point $t=\tau$.
- **Observation 8** - The impulse response of the system has a circular symmetric property with respect to its arguments t and τ .

$$y_{\delta}(t;\tau) = y_{\delta}(\tau,t)$$

Observing an Impulse Response Function (2)

- **Observation 9** – In the (t,τ) -plane, due to the circular symmetry and because delta function is an even function, we can define a bivariate response function as:

$$y(t; \tau) = h(t, \tau)x(\tau, t)\delta(|t - \tau|)$$

- **Observation 10** - The ordinary output response at the point $t=\tau$ is:

$$y(\tau) = \int_{0_-}^{+\infty} h(t, \tau)x(\tau, t)\delta(|t - \tau|)dt = h(\tau)x(\tau)$$

or, equivalently:

$$y(t) = \int_{0_-}^{+\infty} h(t, \tau)x(\tau, t)\delta(|\tau - t|)d\tau = h(t)x(t)$$

Question – Can a system function be equal to an input function?
Is the system output considered to be still linear?

Frequency-Domain Characterization

- A linear time-invariant dynamic system described by a homogeneous n^{th} -order differential equation:

$$\sum_{i=0}^n a_i \frac{d^i}{dt^i} y(t) = 0$$

- Using the operator $d/dt \rightarrow s$, this equation can be written as:

$$\sum_{i=0}^n a_i s^i y(t) = 0$$

- Thus, the homogenous solution is a linear combination of exponentials:

$$y(t) = \sum_{i=0}^n a_i e^{-s_i t}$$

Roots of Characteristics
Operator Equation



Frequency-Domain Characterization (Cont.)

- A linear time-invariant dynamic system with the applied input $x(t)$:

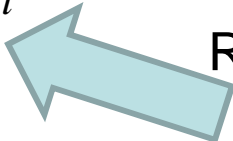
$$\sum_{i=0}^n a_i \frac{d^i}{dt^i} y(t) = \sum_{k=0}^m b_k \frac{d^k}{dt^k} x(t)$$

- The system is turned on at t , the impulse-response at $t-\tau$ is obtained:

$$\sum_{i=0}^n a_i \frac{d^i}{dt^i} y(t; \tau) = \sum_{k=0}^m b_k \frac{d^k}{dt^k} \delta(t - \tau)$$

- We observe:
 - The response is a bivariate function of t and τ
 - Characteristics roots are a function of τ

$$y(t, \tau) = \sum_{i=0}^n a_i e^{-s_i(\tau)t}$$


 Roots of Characteristics
Operator Equation

Laplace Transform of the Impulse Function

- The ordinary unilateral Laplace transform of $\delta(t-\tau)$ is obtained as:

$$L\{\delta(t-\tau)\} = \int_{0-}^{+\infty} \delta(t-\tau) e^{-s_1 t} dt = e^{-s_1 \tau}$$

- This is a function of the variable application time τ .
- A second transformation yields:

$$L_{2D}\{\delta(t-\tau)\} = \int_{0-}^{+\infty} e^{-s_1 \tau} e^{-s_2 \tau} d\tau = \frac{1}{s_1 + s_2}$$



2DLT

Laplace-Carson Transform

Definition of 2DLT – The ordinary unilateral 2DLT is defined as:

$$H(s_1, s_2) = \int_0^{+\infty} \int_0^{+\infty} h(t_1, t_2) e^{-s_1 t_1} e^{-s_2 t_2} dt_1 dt_2$$

Inverse Transformation - The inverse 2DLT is given by:

$$\begin{aligned} L_{2D}^{-1} \{ H(s_1, s_2) \} &= h(t_1, t_2) \\ &= \frac{1}{(2\pi j)^2} \int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} H(s_1, s_2) e^{s_1 t_1} e^{s_2 t_2} ds_1 ds_2 \end{aligned}$$

Two-Dimensional Laplace Transform (2DLT)

Observation 11– For conformal transformation, it is required that the unit function $u(t, \tau)$ transforms into itself:

$$u(t, \tau) \Leftrightarrow U(s_1, s_2) = 1$$

$u(t, \tau)$ is equal to 1 when both t and τ are positive, and is equal to zero when at least one of the arguments is negative.

Observation 12– Based on the above observation we modify the 2DLT is given by:

$$h(t, \tau) \Leftrightarrow H(s_1, s_2) = s_1 s_2 \int_0^{+\infty} \int_0^{+\infty} h(t, \tau) e^{-s_1 t} e^{-s_2 \tau} dt d\tau$$



Laplace-Carson Transform

Two-Dimensional Step Function

□ The unit-step function $u(t, \tau)$ is defined as:

$$u(t, \tau) = u(t)u(\tau)$$

The Laplace-Carson transform of $u(t-\tau)$ is

$$U(s_1, s_2) = s_1 s_2 \int_0^{+\infty} \int_0^{+\infty} u(t-\tau) e^{-s_1 t} e^{-s_2 \tau} dt d\tau = \frac{s_2}{s_1 + s_2}$$

Similarly, the L-C transform of $u(\tau-t)$ is $\frac{s_1}{s_1 + s_2}$

Observation 13 - We can write:

$$u(t-\tau)u(\tau-t) = \begin{cases} 1 & t = \tau \\ 0 & t \neq \tau \end{cases}$$

The 2DLT transform of $u(t-\tau)u(\tau-t)$ is $\frac{1}{s^2}$.

Two-Dimensional Impulse Function

□ The unit-impulse function $\delta(t, \tau)$ is defined as:

$$\delta(t, \tau) = \delta(t)\delta(\tau)$$

The Laplace-Carson transform of $\delta(t, \tau)$ is

$$\Delta(s_1, s_2) = s_1 s_2 \int_0^{+\infty} \int_0^{+\infty} \delta(t - \tau) e^{-s_1 t} e^{-s_2 \tau} dt d\tau = \frac{s_1 s_2}{s_1 + s_2}$$

Similarly, the L-C transform of $\delta(t, \tau)$ is $\delta(t, \tau) \Leftrightarrow s_1 s_2$

Observation 14 - We can obtain the L-C transform of $h(t)\delta(t - \tau)$ as:

$$h(t)\delta(t - \tau) \Leftrightarrow s_1 s_2 \int_0^{+\infty} h(\tau) e^{-(s_1 + s_2)\tau} d\tau = s_1 s_2 H(s_1 + s_2)$$

Impulse-Response System Representation

- The system response for a circularly symmetric system function can equivalently be written as:

$$y_{\delta}(t, \tau) = h(t, \tau)\delta(t - \tau)$$

- The more familiar impulse response, using *sifting property* of the *delta function* will be:

$$y_{\delta}(\tau) = \int_{t=\tau_-}^{t=\tau_+} h(t, \tau)\delta(t - \tau)dt = h(\tau)$$

- The limits of integration can be extended to infinity.

Nonanticipative System Function

- ❖ Define the instant at which the input is applied to the system as the origin for time “*t*.”
- ❖ The **nonanticipative** condition implies:

$$h(t - \tau)x(t)u(t) \equiv 0 \quad \text{for } t < \tau$$

$$h(t - \tau)x(t)u(t) \equiv y(t, \tau) \quad \text{for } t > \tau$$

- ❖ Then, we may define $h(\cdot)$ to be zero for negative values of its argument:

Frequency-Domain Representation of Nonanticipative System Functions

❖ Let us define:

$$h_1(t, \tau) = \begin{cases} h(t - \tau) & \text{for } t > \tau \\ 0 & \text{for } t < \tau \end{cases}$$

❖ The **2DLT** is:

$$H_1(s_1, s_2) = \int_0^{+\infty} e^{-s_2 \tau} d\tau \int_{\tau}^{+\infty} e^{-s_1 t} h(t - \tau) dt = \frac{H(s_1)}{s_1 + s_2}$$


❖ Similarly, we define :

$$h_2(t, \tau) = \begin{cases} h(\tau - t) & \text{for } \tau > t \\ 0 & \text{for } \tau < t \end{cases}$$

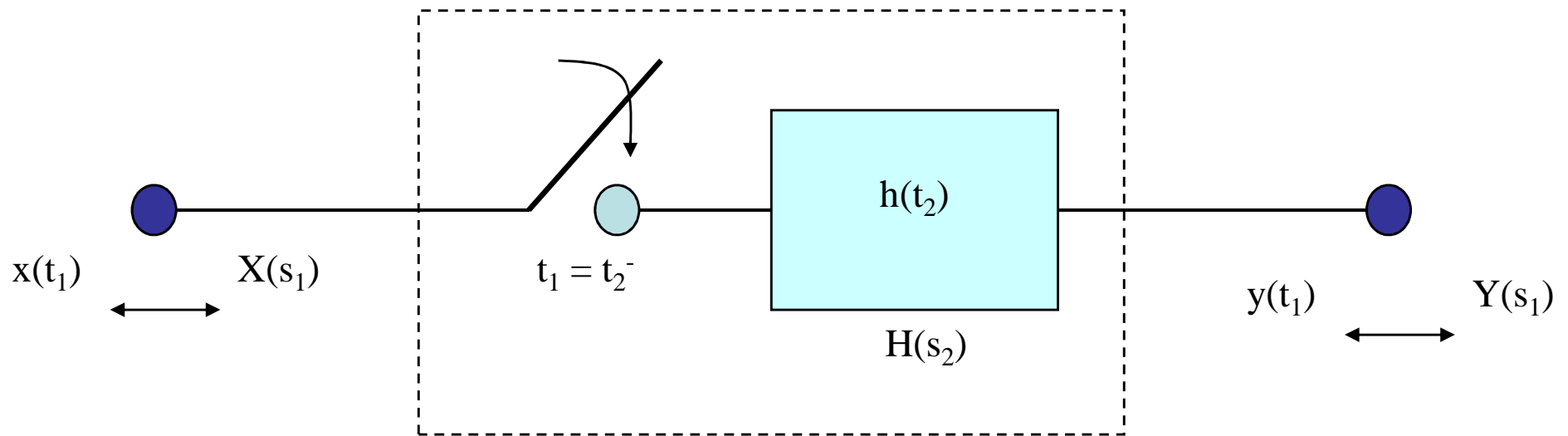
$$H_2(s_1, s_2) = \frac{H(s_2)}{s_1 + s_2}$$

2DLT

❖ Adding together, we obtain:

$$L_{2D} \{h(|t - \tau|)\} = H(s_1, s_2) = \frac{H(s_1) + H(s_2)}{s_1 + s_2}$$


Symbolic Bifrequency Input-Output System Representation



Black box representation of circularly symmetric linear systems

The 2DLT of General LTV Systems

Consider a SISO LTV system, initially at rest, described by:

$$\sum_{i=0}^n a_i(t) \frac{d^i y(t)}{dt^i} = \sum_{k=0}^m b_k(t) \frac{d^k x(t)}{dt^k}$$

An input $x(\cdot)$ is applied to the system at time $\xi = t - \tau$

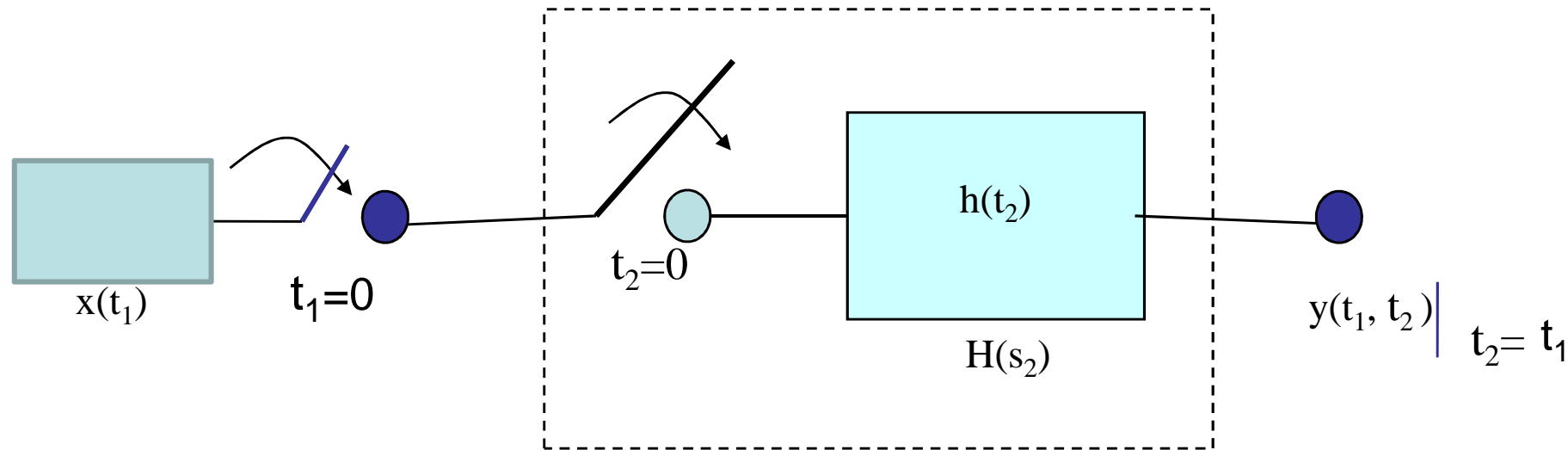
$$\sum_{i=0}^n a_i(t) \frac{d^i y(t, \tau)}{dt^i} = \sum_{k=0}^m b_k(t) \frac{d^k x(t - \tau)}{dt^k}$$

Taking a 2DLT, we obtain:

$$\sum_{i=0}^n \int_0^{\infty} \int_0^{\infty} a_i(t) \frac{d^i y(t, \tau)}{dt^i} e^{-s_2 t} e^{-s_1 \tau} dt d\tau = \sum_{k=0}^m \int_0^{\infty} \int_{\xi=\tau}^t b_k(t) \frac{d^k x(t - \tau)}{dt^k} e^{-s_2 t} e^{-s_1 \tau} dt d\tau$$

This may demand more initial conditions that the problem requires!

Synchronized System Representation



t_1 is the observation time of signal and t_2 is the application time to the system.

The 2DLT of Synchronized LTV Systems

- Consider a SISO LTV system described by:

$$\sum_{i=0}^n a_i(\tau) \frac{d^i y(t)}{dt^i} = \sum_{k=0}^m b_k(\tau) \frac{d^k x(t)}{dt^k}$$

- Taking Laplace transform with respect to τ , we obtain:

$$\sum_{i=0}^n \int_{-\infty}^{+\infty} a_i(\tau) e^{-s_2 \tau} \frac{d^i y(t)}{dt^i} d\tau = \sum_{k=0}^m \int_{-\infty}^{+\infty} b_k(\tau) e^{-s_2 \tau} \frac{d^k x(t)}{dt^k} d\tau$$

$$\sum_{i=0}^n A_i(s_2) \frac{d^i y(t)}{dt^i} = \sum_{k=0}^m B_k(s_2) \frac{d^k x(t)}{dt^k}$$

The 2DLT of Synchronized LTV Systems (Cont.)

- Taking a second transform with respect to t .

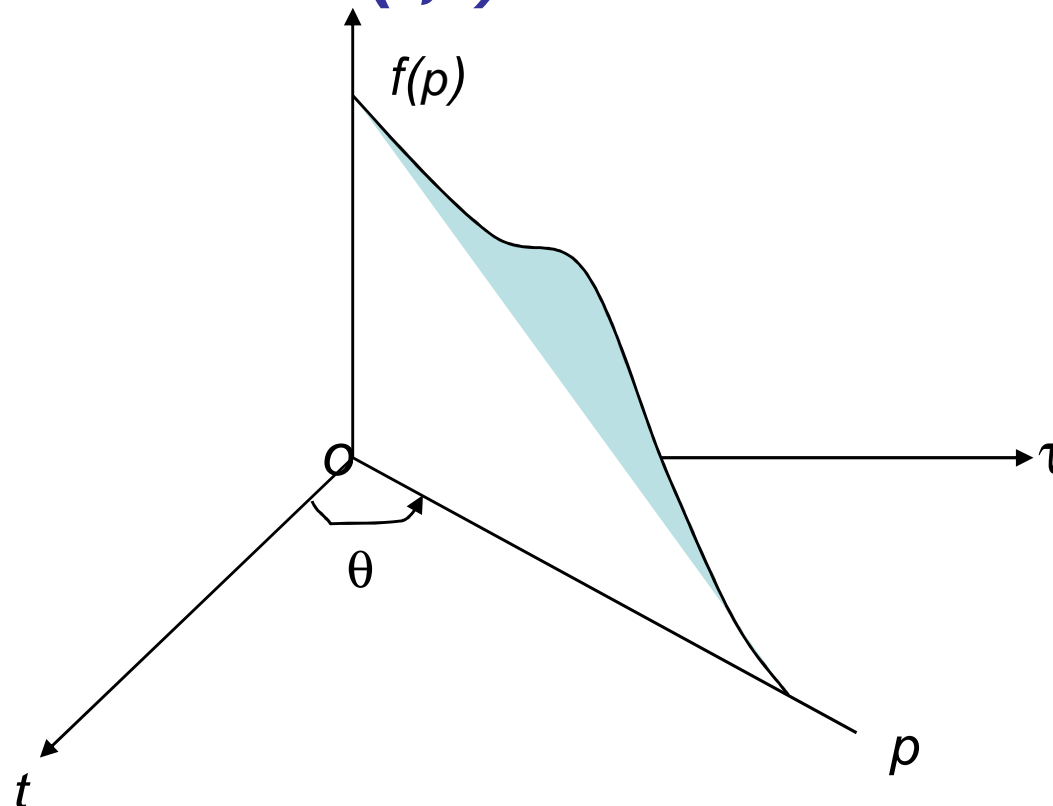
$$\sum_{i=0}^n A_i(s_2) \int_{-\infty}^{+\infty} \frac{d^i y(t)}{dt^i} e^{-s_1 t} dt = \sum_{k=0}^m B_k(s_2) \frac{d^k x(t)}{dt^k} e^{-s_1 t} dt$$

- If the system is initially at rest then $\frac{d^i y(t)}{dt^i} = 0$ for $i=0, 1, 2, \dots, n-1$.

$$\begin{aligned} \sum_{i=0}^n A_i(s_2) s_1^i Y(s_1) &= \sum_{k=0}^m B_k(s_2) \left[s_1^k X(s_1) - s_1^{k-1} x(0) - s_1^{k-2} x^{(1)}(0) - \dots - x^{(k)}(0) \right] = \\ &= \sum_{k=0}^m B_k(s_2) \left[s_1^k X(s_1) - \sum_{j=0}^{k-1} s_1^{k-j} x^{(j)}(0) \right] \end{aligned}$$

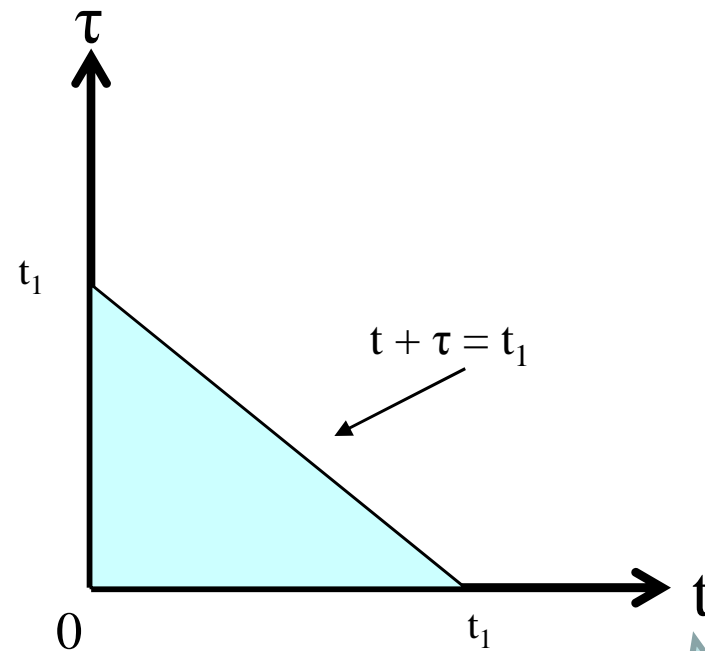
- Thus
$$Y(s_1) = \sum_{k=0}^m B_k(s_2) \left[s_1^k X(s_1) - \sum_{j=0}^{k-1} s_1^{k-j} x^{(j)}(0) \right] \bigg/ \sum_{i=0}^n A_i(s_2) s_1^i$$

Real-Variable Function Representation in (t, τ) -Plane



- A system function, which has the simple rotational property of circular symmetry is shown in this figure.
- Figure shows conversion of a one-dimensional profile of a system function of p to a two-dimensional function of a complex variable of t and τ .
- The common views of the independent signal and system functions are the projection over t -axis (pure real) and τ -axis (pure imaginary), respectively.

Circularly Symmetric System Transformation

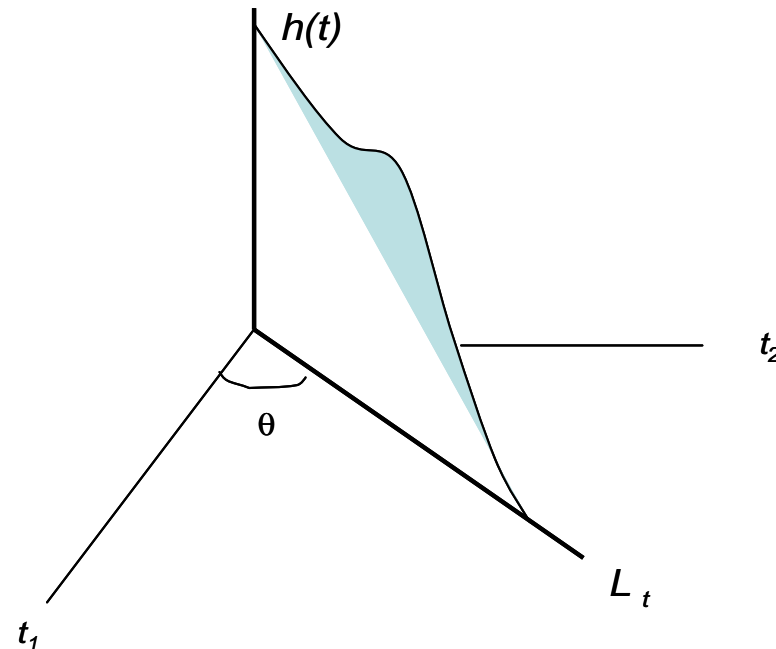


Iterated Laplace Transform

$$Y(s) = \iint_{\Delta} h(t, \tau) x(\tau) e^{-s(t+\tau)} dt d\tau$$

Region over which the impulse-response $h(t, \tau)$ of a nonanticipative system is defined is the shaded area.

Circularly Symmetric Functions (1)



$$h(t, \theta) = h(t \cos \theta, t \sin \theta) \equiv h_{sym}(t) = \frac{1}{2} [h(t_1, t_2) + h(t_2, t_1)]$$

$$H(s, \phi) \equiv H_{sym}(s) = \frac{1}{2} [H(s_1, s_2) + H(s_2, s_1)]$$

Rotational property of a circularly symmetric function

Other Observations on LTV Systems

- Mathematically speaking, t and τ represent time variables of an applied signal and the corresponding system.
- Variations of an input signal and an autonomous system are independent of each other.
- For circularly symmetrical systems, with no loss of generality, we can rewrite the response as:

$$y(t, \tau) = e^{-\ln h(t, \tau)} x(t)$$

Generalized-Delay System Representation

- If $h(t)$ is a (piecewise) continuous function and bounded by a finite number, its 1st-order and higher-order derivatives exists.
- The system response can equivalently be written as:

$$y(t) = e^{-\int_{\tau}^t \frac{h'(\xi)d\xi}{h(\xi)}} x(t)$$

- The system response can be written more compactly as a generalized-delay operator:

$$y(t) = e^{-g(t,\tau)} x(t)$$

The Hankel Transform

- The Hankel transform is compatible with LTV systems described by a general **Bessel** equation given as:

$$\left[\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt} - \left(\frac{n^2}{t^2} \right) \pm a^2 \right]^N y(t) = x(t)$$

- The Hankel transform pairs are symmetric because it deals with symmetric functions.
- The 2DLT of a circularly symmetric function with the property $\lim_{t \rightarrow \infty} h(t) \rightarrow 0$ that is a **Hankel** transform of order zero.
- This property is quite useful in application of Hankel transforms to LTV systems.

The Mellin Transform

- The Mellin transform is compatible with LTV systems characterized by a general **Euler-Cauchy** equation given as:

$$\sum_{i=0}^n a_i t^i \frac{d^i y(t)}{dt^i} = x(t)$$

- The impulse response of this nonanticipative Euler-Cauchy LTV system is:

$$h(t, \tau) = \frac{1}{t} g\left(\frac{t}{\tau}\right) u(t - \tau)$$

- where $g(\cdot)$ is the impulse response of a prototype LTI system obtained by changing the time scale $t \rightarrow -\ln t$

- The 2DLT in this case becomes the following Mellin Transform pairs:

$$M\{h(t)\} = \int_0^{+\infty} h(t) t^{s-1} dt$$

$$M^{-1}\{H(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} H(s) t^{-s} ds$$

Conclusions

- The 2DLT techniques are applicable to LTV systems.
- This investigation justifies application of 2DLT as an operational calculus for system characterization, especially for analog signal processing problems.
- This approach allows, in effect, two-dimensional transform techniques to be used for the time-varying systems in the same manner that the conventional frequency-domain techniques are used in connection with fixed systems.
- The 2DLT, Mellin transform, and Hankel transform can be derived from the **two-dimensional Fourier transform**.
- The work presented here opens several areas for further investigations in theory of variable systems.

For Further Information

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Questions?

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